Differential Geometry and its Applications 27 (2009) 250–256

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Nearly Kähler manifolds with vanishing Tricerri–Vanhecke Bochner curvature tensor

Y. Euh [a], J.H. Park [a,*,1], K. Sekigawa [b]

[a] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea
[b] Department of Mathematics, Niigata University, Niigata 950-2181, Japan

ARTICLE INFO

Article history:
Received 10 October 2007
Received in revised form 21 December 2007
Available online 21 November 2008
Communicated by O. Kowalski

MSC:
53C15
53C55

Keywords:
Curvature tensor
Nearly Kähler manifold
Conformally flat manifold
Bochner flat manifold

ABSTRACT

We study the local structures of nearly Kähler manifolds with vanishing Bochner curvature tensor as defined by Tricerri and Vanhecke (TV). We show that there does not exist a TV Bochner flat strict nearly Kähler manifold in $2n(\geqslant 10)$ dimension and determine the local structures of the manifolds in 6 and 8 dimensions.

1. Introduction

In almost Hermitian geometry, it is both natural and interesting to discuss the relationship between the almost Hermitian structure and given curvature conditions. The Bochner curvature tensor B was defined as a formal analogy of the Weyl conformal tensor [1]. The Bochner flat Kähler manifolds (also known as Bochner–Kähler manifolds) have been discussed by several authors [2,10]. Tricerri and Vanhecke [16] studied the decomposition of the space of all curvature tensors on a Hermitian vector space from the view point of unitary representation theory, and defined the Bochner type curvature tensor $B(R)$ for any almost Hermitian manifold. We call this Bochner type curvature tensor $B(R)$ a Tricerri–Vanhecke (briefly, TV) Bochner curvature tensor. By the definition, the tensor $B(R)$ is invariant under conformal change of the Riemannian metric g. An almost Hermitian manifold with vanishing TV Bochner curvature tensor is called a TV Bochner flat one. Here, we may note that a conformally flat almost Hermitian manifold is necessarily a TV Bochner flat one. It is well-known that a $2n(\geqslant 6)$-dimensional conformally flat Kähler manifold is locally flat [18], and the further local structures of 4-dimensional conformally flat Kähler manifolds have been classified by Tanno [14]. Complex space forms are typical examples of Bochner flat Kähler manifolds. Kamishima [10] has classified all compact Bochner flat Kähler manifolds. On one hand, Gray defined the notion of nearly Kähler manifolds through his research on weak holonomy $U(n)$-structure, which is one of the natural generalizations of Kähler manifolds. Topics in nearly Kähler geometry have been discussed by many authors, for example,

* Corresponding author.
E-mail addresses: prettyfish@skku.edu (Y. Euh), parkj@skku.edu (J.H. Park), sekigawa@math.sc.niigata-u.ac.jp (K. Sekigawa).

1 This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (R01-2008-000-20370-0).

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2008.10.007

Gray [6,8], Moroianu et al. [12] and Nagy [13] (for the more details, refer to Section 3). In the previous paper [5], we studied curvature properties of some special class of 4-dimensional TV Bochner flat almost Hermitian manifolds. The TV Bochner curvature tensor coincides with the Bochner curvature tensor if the manifold is a Kähler manifold [16]. A non-Kähler nearly Kähler manifold is called a strict nearly Kähler manifold. From the above mentioned overviews, the following question will naturally arise:

Question. Classify the local structures of TV Bochner flat (or especially conformally flat) non-Kähler almost Hermitian manifolds.

In the present paper, focusing on the class of nearly Kähler manifolds, we shall prove the following theorem related to the above question.

Main Theorem. *Let $M = (M, J, g)$ be a $2n(\geqslant 6)$-dimensional TV Bochner flat strict nearly Kähler manifold. Then M is either a 6-dimensional space of positive constant sectional curvature or locally a product of a 6-dimensional strict nearly Kähler manifold of positive constant sectional curvature K and a (real) 2-dimensional Kähler manifold of constant Gaussian curvature $-K$.*

Remark. As the result, a $2n(\geqslant 6)$-dimensional TV Bochner flat strict nearly Kähler manifold is a locally symmetric conformally flat space.

From the Main Theorem, taking account of the above remark, we immediately have the following result.

Corollary. *Let $M = (M, J, g)$ be a $2n(\geqslant 6)$-dimensional conformally flat nearly Kähler manifold. Then M is locally either a flat Kähler manifold or a 6-dimensional strict nearly Kähler manifold of positive constant sectional curvature or a product of a 6-dimensional strict nearly Kähler manifold of positive constant sectional curvature K and a (real) 2-dimensional Kähler manifold of constant Gaussian curvature $-K$.*

2. Preliminaries

Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold and Ω the Kähler form of M defined by $\Omega(X, Y) = g(JX, Y)$, for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields X, Y on M. We denote by ∇ and R the Levi-Civita connection and the curvature tensor of (M, J, g) defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{2.1}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Further, we denote by ρ, ρ^*, τ and τ^* the Ricci tensor, the Ricci $*$-tensor, the scalar curvature and the $*$-scalar curvature defined respectively by

$$\begin{aligned} &\rho(X,Y) = \operatorname{tr}\big(Z \longmapsto R(Z,X)Y\big),\\ &\rho^*(X,Y) = \operatorname{tr}\big(Z \longmapsto R(X,JZ)JY\big),\\ &\tau = \operatorname{tr} Q, \qquad \tau^* = \operatorname{tr} Q^* \end{aligned} \tag{2.2}$$

where Q and Q^* are the Ricci operator and the Ricci $*$-operator defined by $g(QX, Y) = \rho(X, Y)$ and $g(Q^*X, Y) = \rho^*(X, Y)$, for $X, Y \in \mathfrak{X}(M)$, respectively. We may easily check that $\rho^*(X, Y) = \rho^*(JY, JX)$ holds for all $X, Y \in \mathfrak{X}(M)$, and $\rho^* = \rho$ holds if M is a Kähler manifold. An almost Hermitian manifold M is called a weakly $*$-Einstein manifold if $\rho^* = \frac{\tau^*}{2n} g$ holds on M and also called a $*$-Einstein manifold especially if τ^* is constant.

Let $\{e_i\}$ be an orthonormal basis of T_pM at any point $p \in M$. In this paper, we shall adopt the following notational convention:

$$\begin{aligned} &R_{ijkl} = g\big(R(e_i,e_j)e_k,e_l\big),\\ &R_{\bar{i}jkl} = g\big(R(Je_i,e_j)e_k,e_l\big),\\ &\dots\\ &R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g\big(R(Je_i,Je_j)Je_k,Je_l\big),\\ &\rho_{ij} = \rho(e_i,e_j), \quad \dots, \quad \rho_{\bar{i}\bar{j}} = \rho(Je_i,Je_j),\\ &\rho^*_{ij} = \rho^*(e_i,e_j), \quad \dots, \quad \rho^*_{\bar{i}\bar{j}} = \rho^*(Je_i,Je_j),\\ &J_{ij} = g(Je_i,e_j), \qquad \nabla_i J_{jk} = g\big((\nabla_{e_i}J)e_j,e_k\big), \end{aligned} \tag{2.3}$$

and so on, where the Latin indices run over the range $1, 2, \dots, 2n$.

The Bochner curvature tensor $B(R)$ is stated below:

$$\begin{aligned} B(R) = R &- \frac{1}{4(n+2)(n-2)} g \triangle \rho + \frac{2n-3}{4(n-1)(n-2)} g \owedge \rho \\ &- \frac{1}{4(n+2)(n-2)} g \triangle (\rho J) + \frac{1}{4(n-1)(n-2)} g \owedge (\rho J) \\ &+ \frac{2n^2-5}{4(n+1)(n+2)(n-2)} g \triangle \rho^* - \frac{2n-1}{4(n+1)(n-2)} g \owedge \rho^* \\ &+ \frac{3}{4(n+1)(n+2)(n-2)} g \triangle \left(\rho^* J\right) - \frac{3}{4(n+1)(n-2)} g \owedge \left(\rho^* J\right) \\ &+ \frac{3n\tau - (2n^2-3n+4)\tau^*}{16(n+1)(n+2)(n-1)(n-2)} g \triangle g - \frac{\tau - \tau^*}{8(n-1)(n-2)} g \owedge g \end{aligned} \tag{2.4}$$

for $n \geqslant 3$, and

$$\begin{aligned} B(R) = R &+ \frac{1}{2} g \owedge \rho + \frac{1}{12} \left\{ g \triangle \rho^* - g \owedge \rho^* - g \triangle \left(\rho^* J\right) + g \owedge \left(\rho^* J\right) \right\} \\ &+ \frac{3\tau^* - \tau}{96} g \triangle g - \frac{\tau + \tau^*}{16} g \owedge g \end{aligned} \tag{2.5}$$

for $n = 2$, where for any $(0,2)$-tensors a and b, we set

$$(a \owedge b)(x, y, z, w) = a(x,z)b(y,w) - a(x,w)b(y,z) + b(x,z)a(y,w) - b(x,w)a(y,z), \tag{2.6}$$

$$\bar{a}(x, y) = a(x, Jy), \tag{2.7}$$

for $x, y, z, w \in T_pM$, $p \in M$, and we set

$$a \triangle b = a \owedge b + \bar{a} \owedge \bar{b} + 2\bar{a} \otimes \bar{b} + 2\bar{b} \otimes \bar{a}. \tag{2.8}$$

The following is the higher-dimensional version of the result in [5].

Theorem 2.1. *Let $M = (M, J, g)$ be a TV Bochner flat almost Hermitian manifold. Then, the curvature tensor R satisfies the following curvature identity*

$$\begin{aligned} &R(X,Y,Z,W) - R(JX,JY,Z,W) - R(X,Y,JZ,JW) + R(JX,JY,JZ,JW) \\ &\quad = R(X,JY,Z,JW) + R(X,JY,JZ,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW) \end{aligned} \tag{2.9}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$.

Remark. The curvature tensor of any Hermitian manifold satisfies the curvature identity (2.9) in the above Theorem 2.1. However, the converse is not true in general. In fact, Tricerri and Vanhecke [15] gave an example of a 6-dimensional locally flat almost Hermitian manifold which is not Hermitian.

3. Nearly Kähler manifolds

An almost Hermitian manifold $M = (M, J, g)$ is called a *nearly Kähler* manifold if $(\nabla_X J)Y + (\nabla_Y J)X = 0$ holds for any $X, Y \in \mathfrak{X}(M)$, namely, the Kähler form of M is a Killing tensor of degree 2. It is easily observed that an integrable nearly Kähler manifold is Kähler. It is immediately checked that any four-dimensional nearly Kähler manifold is necessarily Kähler. We remind that a non-Kähler, nearly Kähler manifold is called a *strict nearly Kähler* manifold. It is well known that a 6-dimensional sphere equipped canonical metric is a typical example of a strict nearly Kähler manifold. There are many other examples of strict nearly Kähler manifolds [7].

We have the following curvature identities [6,8,17] on nearly Kähler manifolds.

$$R_{ijkl} - R_{\bar{i}\bar{j}kl} = -\sum_a (\nabla_i J_{ja}) \nabla_k J_{la} = -\sum_a (\nabla_a J_{ij}) \nabla_a J_{kl}, \tag{3.1}$$

$$\sum \left(R_{ijkl} - R_{ij\bar{k}\bar{l}}\right)\left(\rho_{jk} - \rho^*_{jk}\right) = \frac{1}{4} \sum \left(3\rho_{ia} + \rho^*_{ia}\right)\left(\rho_{la} - \rho^*_{la}\right). \tag{3.2}$$

From (3.1) and (3.2), it is immediate that

$$\sum_{a,b}(\nabla_a J_{bi})\nabla_a J_{bj} = \rho_{ij} - \rho^*_{ij}, \tag{3.3}$$

$$\|\nabla J\|^2 = \tau - \tau^*, \tag{3.4}$$

$$R_{ijkl} = R_{\bar{i}\bar{j}\bar{k}\bar{l}}, \tag{3.5}$$

$$\rho_{\bar{i}\bar{j}} = \rho_{ij}, \qquad \rho^*_{\bar{i}\bar{j}} = \rho^*_{ij}, \tag{3.6}$$

$$\sum(\rho_{ij} - \rho^*_{ij})(\rho_{ij} - 5\rho^*_{ij}) = 0. \tag{3.7}$$

We here refer to the following theorem by Nagy [13] for the strict nearly Kähler manifolds.

Theorem 3.1. *Let $M = (M, J, g)$ be a simply connected, complete strict nearly Kähler manifold. Then M is a product manifold of the following nearly Kähler manifolds*:

(1) 6-*dimensional nearly Kähler manifold*;
(2) *homogeneous nearly Kähler manifold*;
(3) *a twistor space over a quaternionic Kähler manifold with positive scalar curvature.*

Now, we review some results concerning the structures of 6-dimensional, 8-dimensional and 10-dimensional nearly Kähler manifolds. Let $M = (M, J, g)$ be a 6-dimensional strict nearly Kähler manifold. Then, addition to the formulas (3.1)–(3.7), the following equalities hold [11].

$$\nabla_i\nabla_j J_{kl} = -\frac{\tau}{30}(g_{ij}J_{kl} - g_{ik}J_{jl} + g_{il}J_{jk}), \tag{3.8}$$

$$\sum(\nabla_a J_{ij})\nabla_a J_{kl} = -\frac{\tau}{30}(g_{jk}g_{il} - g_{ik}g_{jl} - J_{jk}J_{il} + J_{ik}J_{jl}). \tag{3.9}$$

From (3.1) and (3.9), we have

$$\rho_{ij} - \rho^*_{ij} = \frac{2\tau}{15}g_{ij}. \tag{3.10}$$

From (3.7) and (3.10), we have

$$\rho = \frac{\tau}{6}g \quad \left(\text{and hence } \rho J = \frac{\tau}{6}g\right),$$
$$\rho^* = \frac{\tau}{30}g \quad \left(\text{and hence } \rho^* J = \frac{\tau}{30}g\right). \tag{3.11}$$

Thus, from (3.11), we see that a 6-dimensional strict nearly Kähler manifold M is an Einstein and $*$-Einstein manifold with constant positive scalar curvature. We refer to the following local structure theorems due to Gray [8].

Theorem 3.2. *Let $M = (M, J, g)$ be an 8-dimensional strict nearly Kähler manifold. Then M is locally a product of a 6-dimensional strict nearly Kähler manifold and a (real) 2-dimensional Kähler manifold.*

Theorem 3.3. *Let $M = (M, J, g)$ be a 10-dimensional nearly Kähler manifold. Then we have*

(1) *there exist real numbers α and β with $\alpha^2 \geqslant \beta^2$ such that the eigenvalues of $Q - Q^*$ are $4(\alpha^2 + \beta^2)$ with multiplicity 2, $4\alpha^2$ with multiplicity 4, and $4\beta^2$ with multiplicity 4*;
(2) *$Q - Q^*$ is parallel if and only if $\beta = 0$*;
(3) *if $\beta = 0$ and $\alpha \neq 0$, then M is locally a product of M_1 and M_2, where M_1 is a (real) 4-dimensional Kähler manifold and M_2 is a 6-dimensional strict nearly Kähler manifold.*

The proof of the Main Theorem relies on the formula (3.11) and also on Theorems 3.2, 3.3.

4. TV Bochner flat nearly Kähler manifolds

In this section, first of all, we shall give an explicit form for the curvature tensor of a TV Bochner flat nearly Kähler manifold. From (2.4) and (3.6), we have

Lemma 4.1. *Let $M = (M, J, g)$ be a TV Bochner flat nearly Kähler manifold. Then, the curvature tensor R of M can be expressed explicitly by*

$$\begin{aligned}
R(X,Y,Z,W) = {} & \frac{1}{2(n+2)(n-2)}\big\{g(X,Z)\rho(Y,W) - g(X,W)\rho(Y,Z) + g(Y,W)\rho(X,Z) - g(Y,Z)\rho(X,W) \\
& + g(X,JZ)\rho(Y,JW) - g(X,JW)\rho(Y,JZ) + g(Y,JW)\rho(X,JZ) - g(Y,JZ)\rho(X,JW) \\
& + 2g(X,JY)\rho(Z,JW) + 2g(Z,JW)\rho(X,JY)\big\} \\
& - \frac{n-1}{2(n+2)(n-2)}\big\{g(X,Z)\rho^*(Y,W) - g(X,W)\rho^*(Y,Z) + g(Y,W)\rho^*(X,Z) - g(Y,Z)\rho^*(X,W) \\
& + g(X,JZ)\rho^*(Y,JW) - g(X,JW)\rho^*(Y,JZ) + g(Y,JW)\rho^*(X,JZ) - g(Y,JZ)\rho^*(X,JW) \\
& + 2g(X,JY)\rho^*(Z,JW) + 2g(Z,JW)\rho^*(X,JY)\big\} \\
& - \frac{3n\tau - (2n^2-3n+4)\tau^*}{8(n+1)(n+2)(n-1)(n-2)}\big\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(X,JZ)g(Y,JW) \\
& - g(X,JW)g(Y,JZ) + 2g(X,JY)g(Z,JW)\big\} \\
& - \frac{1}{2(n-2)}\big\{g(X,Z)\rho(Y,W) - g(X,W)\rho(Y,Z) + g(Y,W)\rho(X,Z) - g(Y,Z)\rho(X,W)\big\} \\
& + \frac{1}{2(n-2)}\big\{g(X,Z)\rho^*(Y,W) - g(X,W)\rho^*(Y,Z) + g(Y,W)\rho^*(X,Z) - g(Y,Z)\rho^*(X,W)\big\} \\
& + \frac{\tau-\tau^*}{4(n-1)(n-2)}\big\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big\}
\end{aligned} \tag{4.1}$$

for $n \geqslant 3$.

From (3.1) and (3.3), we have

$$\begin{aligned}
\sum(\nabla_a J_{ij})R_{ijkl}\nabla_a J_{kl} &= \frac{1}{2}\sum(\nabla_a J_{ij})(R_{ijkl} - R_{ij\bar{k}\bar{l}})\nabla_a J_{kl} \\
&= -\frac{1}{2}\sum(\nabla_a J_{ij})(\nabla_b J_{ij})(\nabla_b J_{kl})\nabla_a J_{kl} \\
&= -\frac{1}{2}\sum(\nabla_i J_{ja})(\nabla_i J_{jb})(\nabla_k J_{la})\nabla_k J_{lb} \\
&= -\frac{1}{2}\sum\big(\rho_{ab} - \rho^*_{ab}\big)^2 \\
&= -\frac{1}{2}\big\|Q - Q^*\big\|^2.
\end{aligned} \tag{4.2}$$

On one hand, by (4.1), (3.3) and (3.4), we also have

$$\sum(\nabla_a J_{ij})R_{ijkl}\nabla_a J_{kl} = -\frac{2}{n-2}\big\|Q - Q^*\big\|^2 + \frac{(\tau-\tau^*)^2}{2(n-1)(n-2)}. \tag{4.3}$$

Thus, from (4.2) and (4.3), we have

$$-\frac{1}{2}\big\|Q - Q^*\big\|^2 = -\frac{2}{n-2}\big\|Q - Q^*\big\|^2 + \frac{(\tau-\tau^*)^2}{2(n-1)(n-2)},$$

and hence

$$-(n-6)\|Q - Q^*\|^2 = \frac{1}{n-1}\big(\tau - \tau^*\big)^2. \tag{4.4}$$

From (4.4), we immediately have the following result:

Proposition A. *There does not exist a* $2n(\geqslant 12)$*-dimensional TV Bochner flat strict nearly Kähler manifold.*

By the above Proposition A, it suffices to consider the cases where $6 \leqslant 2n \leqslant 10$ for the proof of the Main Theorem.

5. 6- and 8-dimensional TV Bochner flat nearly Kähler manifolds

First, we shall prove the case of the 6-dimensional TV Bochner flat nearly Kähler manifold. Let $M = (M, J, g)$ be a 6-dimensional TV Bochner flat strict nearly Kähler manifold. Then from (4.1), taking account of (3.11), we shall see immediately that

$$R(X,Y,Z,W) = -\frac{\tau}{30}\big\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big\} \tag{5.1}$$

holds for $X, Y, Z, W \in \mathfrak{X}(M)$. Therefore, we see finally that M is a space of positive constant sectional curvature $\frac{\tau}{30}$. This completes the proof of the 6-dimensional case.

Next, we shall prove the case of the 8-dimensional TV Bochner flat nearly Kähler manifold. Let $M = (M, J, g)$ be an 8-dimensional strict nearly Kähler manifold. Then, by Theorem 3.2, M is locally a product of a 6-dimensional strict nearly Kähler manifold $M_1 = (M_1, J_1, g_1)$ and a (real) 2-dimensional Kähler manifold $M_2 = (M_2, J_2, g_2)$. Thus, we see that (J, g) is locally expressed by $J = (J_1, J_2)$ and $g = (g_1, g_2)$. Further, we see that the Ricci tensor ρ and the Ricci $*$-tensor ρ^* are locally expressed by $\rho = (\rho_1, \rho_2)$ and $\rho^* = (\rho_1^*, \rho_2^*)$, respectively, where ρ_1 (resp. ρ_2) and ρ_1^* (resp. ρ_2^*) are the Ricci tensor and the Ricci $*$-tensor of M_1 (resp. M_2), respectively. Since (M_1, J_1, g_1) (resp. (M_2, J_2, g_2)) is a 6-dimensional strict nearly Kähler manifold (resp. (real) 2-dimensional Kähler manifold), we have

$$\rho_1 = \frac{\tau_1}{6} g_1, \quad \rho_1^* = \frac{\tau_1}{30} g_1 \quad \left(\text{resp.} \quad \rho_2 = \frac{\tau_2}{2} g_2, \ \rho_2^* = \frac{\tau_2}{2} g_2\right) \tag{5.2}$$

by virtue of (3.11), where τ_1 (resp. τ_2) is a scalar curvature of M_1 (resp. M_2). Thus we have

$$(\rho_{ij}) = \begin{pmatrix} \frac{\tau_1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\tau_1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\tau_1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\tau_1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\tau_1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tau_1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau_2}{2} \end{pmatrix},$$

$$\left(\rho_{ij}^*\right) = \begin{pmatrix} \frac{\tau_1}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\tau_1}{30} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\tau_1}{30} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\tau_1}{30} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\tau_1}{30} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tau_1}{30} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau_2}{2} \end{pmatrix} \tag{5.3}$$

with respect to a unitary basis $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3, e_5, e_6 = Je_5, e_7, e_8 = Je_7\}$, where $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $\{e_7, e_8\}$ are tangent to M_1 and M_2, respectively. Therefore we have

$$\tau = \tau_1 + \tau_2, \qquad \tau^* = \frac{\tau_1}{5} + \tau_2,$$

and hence

$$\tau_1 = \frac{5}{4}\left(\tau - \tau^*\right), \qquad \tau_2 = \frac{1}{4}\left(5\tau^* - \tau\right). \tag{5.4}$$

Thus, from (4.1), taking account of (5.2) and (5.4), we have

$$0 = R(e_a, Je_a, e_7, Je_7) = \frac{1}{90}\left(\tau - 7\tau^*\right),$$

for $a = 1, 3, 5$ and hence

$$\tau = 7\tau^*. \tag{5.5}$$

Therefore, by (5.2)–(5.5), we see that if $-K$ is the Gaussian curvature of M_2, then $\tau_2 = -2K$, $\tau^* = 4K$, $\tau_1 = 30K$ hold (thus K is a positive constant), and hence

$$(\rho_{ij}) = \begin{pmatrix} 5K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5K & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K \end{pmatrix}.$$

This completes the proof of the 8-dimensional case.

6. 10-dimensional TV Bochner flat nearly Kähler manifold

We shall prove the case of the 10-dimensional TV Bochner flat nearly Kähler manifold. Let $M = (M, J, g)$ be a 10-dimensional TV Bochner flat strict nearly Kähler manifold. Then, by Theorem 3.3, for any point $p \in M$, we may choose a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3, e_5, e_6 = Je_5, e_7, e_8 = Je_7, e_9, e_{10} = Je_9\}$ of T_pM such that

$$\begin{aligned} &(Q - Q^*)e_a = 4(\alpha^2 + \beta^2)e_a, \\ &(Q - Q^*)e_s = 4\alpha^2 e_s, \\ &(Q - Q^*)e_u = 4\beta^2 e_u, \end{aligned} \tag{6.1}$$

for $a = 1, 2$, $s = 3, 4, 5, 6$, $u = 7, 8, 9, 10$. From (6.1), we have easily

$$\tau - \tau^* = \operatorname{trace}(Q - Q^*) = 24(\alpha^2 + \beta^2), \tag{6.2}$$

$$\|Q - Q^*\|^2 = 32(\alpha^2 + \beta^2)^2 + 64(\alpha^4 + \beta^4). \tag{6.3}$$

From (6.2) and (6.3), taking account of (4.4), we have

$$\begin{aligned} &32(\alpha^2 + \beta^2)^2 + 64(\alpha^4 + \beta^4) = 144(\alpha^2 + \beta^2)^2, \\ &7(\alpha^2 + \beta^2)^2 = 4(\alpha^4 + \beta^4), \\ &3\alpha^4 + 3\beta^4 + 14\alpha^2\beta^2 = 0, \end{aligned} \tag{6.4}$$

and hence, it must follow that $\alpha = \beta = 0$. By (3.4) and (6.2), we have $\|\nabla J\|^2 = 0$. Thus M is a Kähler manifold, which is a contradiction. Thus we have the following result:

Proposition B. *There does not exist a* 10*-dimensional TV Bochner flat strict nearly Kähler manifold.*

7. Some remarks

Butruille [3] proved that a 6-dimensional almost Hermitian manifold of type $W_1 + W_4$ in the Gray–Hervella classification [9] has the closed Lee form, and hence, it is a locally conformal nearly Kähler manifold. Combining this result and the one by Cleyton and Ivanov ([4], Lemma 8), we may see that a 6-dimensional almost Hermitian manifold of type $W_1 + W_4$ is a globally conformal nearly Kähler manifold or a locally conformal Kähler manifold, and hence, in particular, a 6-dimensional almost Hermitian manifold of type $W_1 + W_4$ which is not of type W_4 is globally conformal to a strict nearly Kähler manifold. Thus, from the above result and the main theorem of the present paper, we see that a 6-dimensional TV Bochner flat almost Hermitian manifold of type $W_1 + W_4$ is either globally conformal to a strict nearly Kähler manifold of a positive constant sectional curvature or locally conformal to a Bochner flat Kähler manifold. Moroianu and Ornea classified all compact 5-dimensional Riemannian manifolds (M, g) with the property that the Riemannian cylinder $(M \times \mathbb{R}, g + dt^2)$ carries an almost Hermitian structure of type $W_1 + W_4$ ([12], Theorem 5.1).

References

[1] S. Bochner, Curvature and Betti numbers II, Ann. Math. 50 (1949) 77–93.
[2] R.L. Bryant, Bochner–Kähler metrics, J. Amer. Math. Soc. 14 (2001) 623–715.
[3] J.-B. Butruille, Espace de twisteurs réduit d'une variété presque hermitienne de dimension 6, math.DG/0503150, 2005.
[4] R. Cleyton, S. Ivanov, Conformal equivalence between certain geometries in dimension 6 and 7, math.DG/0607487, 2006.
[5] Y. Euh, J. Lee, J.H. Park, K. Sekigawa, A. Yamada, Four-dimensional almost Hermitian manifolds with vanishing Tricerri–Vanhecke Bochner curvature tensor, arXiv: 0708.1403v1 [math.DG].
[6] A. Gray, Nearly Kähler manifolds, J. Differential Geom. 4 (1970) 283–310.
[7] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geom. 7 (1972) 343–369.
[8] A. Gray, The structure of nearly Kähler manifolds, Math. Ann. 223 (1976) 233–248.
[9] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. di Mat. Pura Appl. 123 (1980) 35–58.
[10] Y. Kamishima, Uniformization of Kähler manifolds with vanishing Bochner tensor, Acta Math. 172 (1994) 299–308, Acta Math. 195 (2005) 265–266.
[11] M. Matsumoto, On 6-dimensional almost Tachibana spaces, Tensor N.S. 23 (1972) 250–252.
[12] A. Moroianu, L. Ornea, Conformally Einstein products and nearly Kähler manifolds, Ann. Glob. Anal. Geom. 33 (2008) 11–18.
[13] P. Nagy, On nearly-Kähler geometry and Riemannian foliations, Asian J. Math. 16 (2002) 481–504.
[14] S. Tanno, 4-dimensional conformally flat Kähler manifolds, Tohoku Math. J. 24 (1972) 501–504.
[15] F. Tricerri, L. Vanhecke, An example of a 6-dimensional flat almost Hermitian manifold, Colloq. Math. 50 (1978) 111–117.
[16] F. Tricerri, L. Vanhecke, Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc. 267 (1981) 365–398.
[17] K. Takamatsu, Y. Watanabe, On a K-space of constant holomorphic sectional curvature, Kodai Math. Sem. Rep. 25 (1973) 297–306.
[18] K. Yano, Differential Geometry on Complex and Almost Complex Spaces, Pergamon, New York, 1965.